CYLINDRICAL and SPHERICAL COORDINATES

In the ***cylindrical coordinate system***, a point P in three-dimensional space is represented by the ordered triple (r, θ, z) where r and θ are the <u>polar coordinates</u> of the projection of P onto the <u>xy-plane</u> and z is the directed distance from the xy-plane to P.

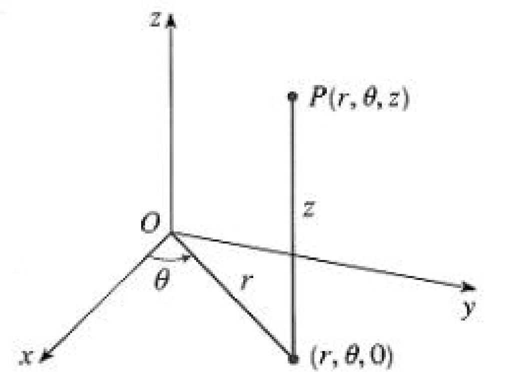

FIGURE 1
The cylindrical coordinates of a point

To convert the point (x, y, z) from rectangular to cylindrical coordinates we use: $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$, and z = z. [See Appendix H.1 on polar coordinates!] The point (3, -3, -7) becomes $\left(3\sqrt{2}, \frac{7\pi}{4}, -7\right)$ in cylindrical coordinates. As with polar coordinates, there are infinitely many choices for θ. Another correct answer would be $\left(3\sqrt{2}, -\frac{\pi}{4}, -7\right)$.

1. Change (3, 3, -2) from rectangular coordinates to cylindrical.

From polar coordinates we recall that $x = r\cos\theta$ and $y = \sin\theta$. We will use these equations to convert from cylindrical to rectangular coordinates. [Remember that z = z]. $\left(2, \frac{2\pi}{3}, 1\right)$ becomes $\left(-1, \sqrt{3}, 1\right)$ in rectangular coordinates.

2. Change $\left(5, \frac{\pi}{6}, 6\right)$ from cylindrical to rectangular coordinates.

A simple substitution using $r^2 = x^2 + y^2$ will convert the [rectangular] equation $4x^2 + 4y^2 + z^2 = 1$ into an equation in cylindrical coordinates: $z^2 = 1 - 4(x^2 + y^2) = 1 - 4r^2$.

3. Write $x^2 + y^2 - z^2 = 16$ in cylindrical coordinates.

If it is not possible to isolate the expression $x^2 + y^2$, we can use the conversion equations for x and y as given above. $x^2 - y^2 = z \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = z \Rightarrow z = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$.

4. Write $x^2 - y^2 - z^2 = 1$ in cylindrical coordinates.

The spherical coordinates (r, q, f) of a point P in space are shown in this diagram, where $r = |OP|$ is the distance from the origin to P, q is the same angle as in cylindrical coordinates, and f is the angle between the positive z-axis and the line segment OP. Note that $r \geq 0$ and $0 \leq f \leq p$. The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

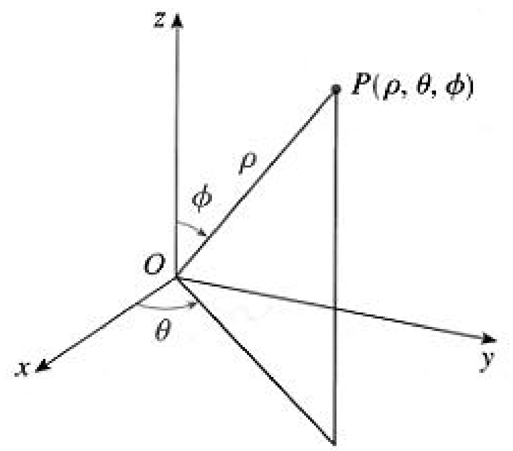

FIGURE 5
The spherical coordinates of a point

To convert from spherical to rectangular coordinates, we use: $x = r \sin f \cos q$, $y = r \sin f \sin q$, and $z = r \cos f$.

$\left(2, \frac{p}{4}, \frac{p}{3}\right)$ becomes $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$.

5. Change $\left(5, \pi, \frac{\pi}{2}\right)$ from spherical coordinates to rectangular.

We may use the distance formula to convert from rectangular to spherical coordinates: $r^2 = x^2 + y^2 + z^2$. Given the point $(0, 2\sqrt{3}, -2)$ in rectangular coordinates: $r^2 = 0 + 12 + 4 \Rightarrow r = 4$. $z = r\cos f \Rightarrow \cos f = \frac{-2}{4} = -\frac{1}{2} \Rightarrow f = \frac{2p}{3}$. $\cos q = \frac{x}{r \sin f} = 0 \Rightarrow q = \frac{p}{2}$. Therefore, $(0, 2\sqrt{3}, -2)$ becomes $\left(4, \frac{p}{2}, \frac{2p}{3}\right)$.

6. Change $\left(1, \sqrt{3}, 2\right)$ from rectangular coordinates to spherical.

We can also use the conversion equations for x, y, and z as given above to change the equation $x^2 - y^2 - z^2 = 1$ from rectangular coordinates to spherical.

$$(r \sin f \cos q)^2 - (r \sin f \sin q)^2 - (r \cos f)^2 = 1$$
$$r^2 \sin^2 f \cos^2 q - r^2 \sin^2 f \sin^2 q - r^2 \cos^2 f = 1$$
$$r^2(\sin^2 f \cos^2 q - \sin^2 f \sin^2 q - \cos^2 f) = 1$$
$$r^2[\sin^2 f(\cos^2 q - \sin^2 q) - \cos^2 f] = 1$$
$$r^2[\sin^2 f \cos 2q - \cos^2 f] = 1$$

7. Write $x^2 + y^2 - z^2 = 16$ in spherical coordinates.

8. Write $2y^2 - 2x^2 = z$ in spherical coordinates.